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Sensitivity analysis and shape optimization for transient heat conduction with radiation

Ryszard Korycki

Department of Technical Mechanics K-411, Technical University of Lodz, Zeromskiego 116, 90-543 Lodz, Poland

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Abstract

Transient heat conduction problem is stated by the differential heat conduction equation, thermal boundary conditions on the external and internal boundary portions and the initial condition within the domain. Next an arbitrary behavioral functional is defined and its first-order sensitivities are determined using the material derivative concept as well as both the direct and adjoint approaches. The most used shape domain modifications are discussed in order to investigate the effect of design parameters on the integral radiation condition. The shape optimization problem is next formulated applying the obtained sensitivities. The illustration is the simple example of the shape optimization.

Keywords: Radiation; Sensitivity analysis; Shape optimization

1. Introduction

Radiative heat transfer is the important fundamental phenomena existing in practical engineering. The examples are the solar radiation in buildings, foundry engineering and solidification processes, die forging, chemical engineering, composite structures applied in industry. The physical analysis demonstrates that the radiative heat transfer problems are encountered as well in textiles (i.e. industrial textiles, textiles designed for use under hermetic protective barrier, multilayer clothing materials, etc.) as in textile structures (i.e. needle heating in heavy industrial sewing). Each of the above-mentioned radiative problems is the particular case characterized by a set of governing equations. Dems and Korycki [4] discuss some of these problems and give a short review of literature. The radiation within the hole is described here by the non-local integral condition according to Bialecki et al. [1,2]. The result is an integral equation describing the radiation intensification (caused by the reflected radiation) and absorbing of the radiation within the isothermal and participating medium. These problems can be solved using different methods (cf. [15]). Roche and Sokolowski [14] gave also more information concerning numerical methods applying in optimization practice.

The presented paper is an extension of the steady problems stated and discussed by Dems and Korycki [4]. Other best general references here can be Dems and Mróz [3], Dems and Rousselet [6,7], and Korycki [10–13]. The first-order sensitivities of an arbitrary behavioral functional will be formulated as a function of the transformation velocity field and solutions of primary, direct and adjoint heat transfer problems, cf. [9].

The aim of this paper is to introduce the first-order sensitivities of an arbitrary behavioral functional to the shape design problems associated with the radiative heat transfer. A much more general modeling of transient conduction problems is considered here in view of radiative heat transfer on both the external and internal boundaries described by different conditions. These problems were not yet considered in the analyzed literature for the integral formulation of the radiation condition in transient problems.

E-mail address: rkorycki@p.lodz.pl

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Nomenclature

a absorption coefficient of the radiation within the medium
A material conductivity matrix
b design parameters vector
c material heat capacity
C structural cost, constraint in the shape optimization problem
C_0 structural cost on the assumed level
$e_b(T) = \sigma T^4$ blackbody emissive power described by the Stefan–Boltzmann law
f heat generation source of the primary structure
f^a heat generation source of the adjoint structure
F optional objective functional
$g_p = Dg/Db_p$ global (material) derivative of the function g with respect to design parameter b_p
$g^p = \partial g/\partial b_p$ local (domain) derivative of the function g with respect to design parameter b_p calculated for the fixed domain Ω
h surface film conductance
H main curvature of the structural boundary Γ
$K(\mathbf{r}, \mathbf{p})$ kernel function of the radiation
n unit vector directed outwards on the boundary Γ
N number of objective functionals
p vector-coordinate of the observation point
P number of design parameters
$q_n = \mathbf{n} \cdot \mathbf{q}$ heat flux density normal to the boundary
$\tilde{q}_n^r = \mathbf{n} \cdot \tilde{\mathbf{q}}^r$ radiative heat flux density normal to the internal boundary Γ_r
q heat flux density vector
$\mathbf{q}^a$ heat flux density vector of the adjoint structure
$\mathbf{q}^*$ initial heat flux density vector
$\mathbf{q}^{*a}$ initial heat flux density vector of the adjoint structure
r vector-coordinate of the current point
t time of the primary and additional problem
T state variable of the primary problem, the temperature field
T^a state variable of the adjoint problem, the temperature field within adjoint structure
T^p state variable of the additional problem associated with design parameter b_p
T_∞ surrounding temperature
T_∞^a adjoint surrounding temperature
T_m temperature measured during the optimization on the boundary portion Γ_m
u unit cost of the material
$\mathbf{v}^p(\mathbf{x}, \mathbf{b}, t)$ transformation velocity field associated with the design parameter b_p
$v_n^p = \mathbf{n} \cdot \mathbf{v}^p$ transformation velocity normal to the boundary
x vector of the coordinates
Γ external boundary surrounding the domain Ω
Γ_T external boundary portion of the prescribed temperature
Γ_q external boundary portion of the prescribed heat flux density
Γ_c external boundary portion of the prescribed convectional heat flux density
Γ_d external boundary portion with the radiation
Γ_r internal boundary with the radiation
ε surface emissivity
σ the Stefan–Boltzmann constant
Σ discontinuity line between two adjacent parts of the piecewise smooth boundary
τ time of the adjoint problem
Φ_p, Φ_r angles between the line of sight and the normal to the surface directed outwards on Γ at the observation and the current points, respectively
$\varphi(\mathbf{x}, \mathbf{b}, t)$ given function of the space $\boldsymbol{x}$, the design parameters $\boldsymbol{b}$, and the time t
χ Lagrange multiplier which is an optional real number
$\boldsymbol{\omega}$ rotation vector characterizing the rotation of the domain
ξ^2 additional variable (slack variable) in Lagrange functional
Ω thermal anisotropic domain of the structure
∇ gradient operator
$(\cdot)_{,\Delta}$ local derivative of the adequate function $(\cdot)$ with respect to Δ
$(\cdot)_p$ global derivative of the adequate function $(\cdot)$ with respect to design parameter b_p

2. Primary heat conduction problem

Let us introduce the transient heat conduction problem within a thermal anisotropic domain Ω bounded by the boundary Γ (Fig. 1). The state variable is now the temperature T. The radiation on the part Γ_d of the external boundary is stated using the Stefan–Boltzmann law. The boundary portion Γ_r is an internal boundary and the radiation can be described using the most general form of radi-

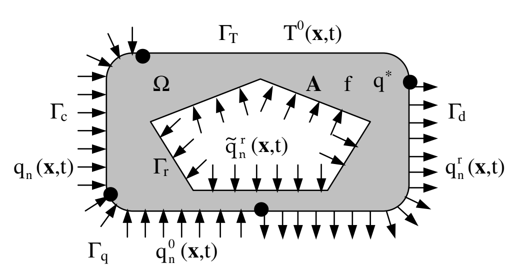

Fig. 1. Primary heat conduction problem.

ation condition. Thus, the heat conduction equation, the boundary conditions as well as the initial condition are given in the form

$$\left.\begin{aligned}&-\operatorname{div}\mathbf{q}+f=c\tfrac{\mathrm{d}T}{\mathrm{d}t}\\&\mathbf{q}=\mathbf{A}\cdot\nabla T+\mathbf{q}^{*}\end{aligned}\right\}\quad \mathbf{x}\in\Omega;$$
$$T(\mathbf{x},t)=T^{0}(\mathbf{x},t)\quad \mathbf{x}\in\Gamma_T;$$
$$q_n^r(\mathbf{x},t)=\sigma T(\mathbf{x},t)^4\quad \mathbf{x}\in\Gamma_d;$$
$$q_n(\mathrm{x},t)=\mathbf{n}\cdot\mathbf{q}(\mathbf{x},t)=q_n^0(\mathbf{x},t)\quad \mathbf{x}\in\Gamma_q;$$
$$\mathbf{n}\cdot\tilde{\mathbf{q}}^r(\mathbf{x},t)=\tilde{q}_n^r(\mathbf{x},t)\quad \mathbf{x}\in\Gamma_r;$$
$$q_n(\mathrm{x},t)=h[T(\mathbf{x},t)-T_\infty(\mathbf{x},t)]\quad \mathbf{x}\in\Gamma_c;$$
$$T(\mathbf{x},t=0)=T_0(\mathbf{x},t=0)\quad \mathbf{x}\in(\Omega\cup\Gamma). \tag{1}$$

To determine the radiative heat flux density $\tilde{q}_n^r$ on the boundary Γ_r of the convex hole we solve the governed radiation condition stated by Bialecki [1], discussed by Dems and Korycki [4]

$$\begin{aligned}&\tilde{q}_n^r(\mathbf{p})+\varepsilon(\mathbf{p})e_\mathrm{b}[T(\mathbf{p})]\\&\quad=\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{e_\mathrm{b}[T(\mathbf{r})]+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}K(\mathbf{r},\mathbf{p})\\&\qquad\times\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r+\varepsilon(\mathbf{p})e_\mathrm{b}(T^m)\\&\qquad\times\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})[1-\exp(-a|\mathbf{r}-\mathbf{p}|)]\,\mathrm{d}\Gamma_r.\end{aligned} \tag{2}$$

Kernel function depends on the geometry of the hole ($|\mathbf{r}-\mathbf{p}|$ is the distance between the current and observation points) and for two-dimensional problems can be expressed according [1]

$$K(\mathbf{r},\mathbf{p})=\frac{\cos\Phi_p\cos\Phi_r}{2|\mathbf{r}-\mathbf{p}|}. \tag{3}$$

The shape variation of domain Ω together with its external boundary Γ due to an infinitesimal transformation process can be defined according [6,7] in the form (cf. Fig. 2)

$$\begin{aligned}\Omega\to\Omega^t:\quad \mathbf{x}^t&=\mathbf{x}+\delta\varphi(\mathbf{x},\mathbf{b},t)=\mathbf{x}+\frac{\partial\varphi(\mathbf{x},\mathbf{b},t)}{\partial b_p}\delta b_p\\&=\mathbf{x}+\mathbf{v}^p(\mathbf{x},\mathbf{b},t)\delta b_p,\end{aligned} \tag{4}$$

where a transformation velocity field is treated as a time-like parameter. Let us state an arbitrary behavioral functional associated with above heat transfer problem, given in the form

$$F=\int_0^{t_\mathrm{f}}\left[\int_{\Omega(\mathbf{b})}\Psi(T,\nabla T,\mathbf{q},f,\dot{T})\,\mathrm{d}\Omega+\int_{\Gamma(\mathbf{b})}\gamma(T,q_n,T_\infty)\,\mathrm{d}\Gamma\right]\mathrm{d}t, \tag{5}$$

where Ψ and γ are continuous and differentiable functions of their parameters. The first-order sensitivity F_p is considered as the material derivative of functional F with respect to the design parameter b_p according [16]. After some transformations we express the sensitivity in the basic form

$$\begin{aligned}F_p&=\frac{\mathrm{D}F}{\mathrm{D}b_p}\\&=\int_0^{t_\mathrm{f}}\int_\Omega[\Psi_{,T}T_p+\nabla_{\nabla T}\Psi(\nabla T)_p+\Psi_{,\mathbf{q}}\cdot\mathbf{q}_p+\Psi_{,\mathrm{f}}f_p\\&\quad+\Psi_{,\dot{T}}\dot{T}_p+\Psi\operatorname{div}\mathbf{v}^p]\,\mathrm{d}\Omega\,\mathrm{d}t+\int_0^{t_\mathrm{f}}\int_\Gamma[\gamma_{,T}\mathrm{T}_p+\gamma_{,q_n}(q_n)_p\\&\quad+\gamma_{,T_\infty}(T_\infty)_p+\gamma(\operatorname{div}_\Gamma\mathbf{v}^p-2Hv_n^p)]\,\mathrm{d}\Gamma\,\mathrm{d}t,\end{aligned} \tag{6}$$

where $\nabla_{\nabla T}\Psi=\left|\frac{\partial\Psi}{\partial T_{,1}};\frac{\partial\Psi}{\partial T_{,2}};\frac{\partial\Psi}{\partial T_{,3}}\right|$. The unknown sensitivities of state fields in Eq. (6) can be derived using the direct approach to sensitivity analysis or can be eliminated from Eq. (6) by means of adjoint state fields, obtaining as the result of solution of adjoint heat transfer problem, alternatively.

3. Direct approach to sensitivity analysis

The direct approach is most useful for calculating the sensitivities of entire response field with respect to a few design variables (Fig. 3). Dems and Rousselet [6,7], Dems et al. [5] and Korycki [10–13] analyzed this approach for example. The solutions of the additional heat conduction problems associated with variation of each design parameter b_p are now necessary to obtain the first-order sensitivities of the functional Eq. (5). The additional structure has the same shape and thermal properties as the primary one and is characterized by the heat conduction equation, boundary and initial conditions, respectively. Above conditions are stated by differentiation of primary equations with respect to design parameters.

The governing equations for the additional structure are formulated applying Dems et al. [5], Dems and Rousselet [6], as well as by using Eq. (1) in the following final form:

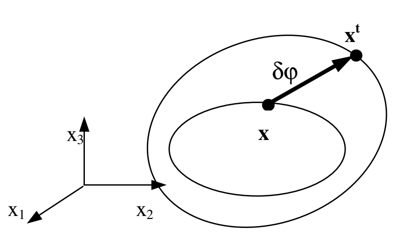

Fig. 2. Infinitesimal transformation of the domain Ω with its external boundary Γ.

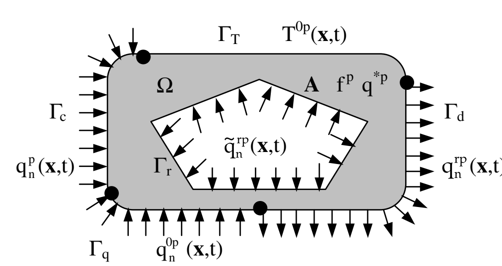

Fig. 3. Additional heat conduction problem.

$$\begin{aligned}
&\left.\begin{array}{l} -\mathrm{div}\,\mathbf{q}^p + f^p = c\frac{\mathrm{d}T^p}{\mathrm{d}t} \\ \mathbf{q}^p = \mathbf{A}\cdot\nabla T^p + \mathbf{q}^{*p} \end{array}\right\} \quad \mathbf{x}\in\Omega;\\
&q_n^p(\mathbf{x},t) = (q_n^0)_p + \mathbf{q}_\Gamma^0\cdot\nabla_\Gamma v_n^p - \nabla_\Gamma q_n^0\cdot\mathbf{v}_\Gamma^p - q_{n,n}^0 v_n^p \quad \mathbf{x}\in\Gamma_q;\\
&T^p(\mathbf{x},t) = T^{0p} = T_p^0 - \nabla T^0\cdot\mathbf{v}^p \quad \mathbf{x}\in\Gamma_T;\\
&q_n^p(\mathbf{x},t) = \mathbf{n}\cdot\mathbf{q}^p = h(T^p - T_\infty^p) + \mathbf{q}_\Gamma\cdot\nabla_\Gamma v_n^p \quad \mathbf{x}\in\Gamma_c;\\
&q_n^{rp} = 4\sigma T^3(T^p + \nabla T\cdot\mathbf{v}^p) - \nabla q_n^r\cdot\mathbf{v}^p + \mathbf{q}^r\cdot\nabla_\Gamma v_n^p \quad \mathbf{x}\in\Gamma_d;\\
&\mathbf{n}\cdot\tilde{\mathbf{q}}^{rp} = \tilde{q}_n^{rp}\mathbf{x}\in\Gamma_r;\, T_0^p(\mathbf{x},0) = T_{0p} - \nabla T_0\cdot\mathbf{v}^p \quad \mathbf{x}\in(\Omega\cup\Gamma).
\end{aligned} \tag{7}$$

In order to calculate the radiative heat flux density $\tilde{q}_n^{rp}$ on the boundary Γ_r of the convex hole, we should formulate the radiation condition. Let us differentiate the condition Eq. (2) for primary problem using the material derivative concept. Our procedure starts with the left-hand side of Eq. (2), denoted by (L). Introducing Eq. (7) and the Stefan–Boltzmann law we can determine the following condition:

$$\begin{aligned}
\frac{\mathrm{D}L}{\mathrm{D}b_p} &= \frac{\mathrm{D}}{\mathrm{D}b_p}[\tilde{q}_n^r(\mathbf{p})] + \frac{\mathrm{D}}{\mathrm{D}b_p}[\varepsilon(\mathbf{p})]e_\mathrm{b}[T(\mathbf{p})] + \varepsilon(\mathbf{p})\frac{\mathrm{D}}{\mathrm{D}b_p}[e_\mathrm{b}[T(\mathbf{p})]]\\
&= \mathbf{n}\cdot(\tilde{q}^r)_p + \sigma[T(\mathbf{p})]^4\varepsilon_p(\mathbf{p}) + 4\sigma T(\mathbf{p})^3\varepsilon(\mathbf{p})[T(\mathbf{p})]_p\\
&= (\tilde{q}_n^r)^p - \tilde{q}_\Gamma^r\cdot\nabla_\Gamma v_n^p + \nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p + \tilde{q}_{n,n}^r v_n^p\\
&\quad + \sigma T(\mathbf{p})^3[T(\mathbf{p})\varepsilon_p(\mathbf{p}) + 4T^p(\mathbf{p}) + 4\nabla T(\mathbf{p})\cdot\mathbf{v}^p].
\end{aligned} \tag{8}$$

Our next objective is to determine the material derivative of the right-hand side of Eq. (2), denoted by (R). For the constant value of absorption coefficient it has the following form:

$$\begin{aligned}
\frac{\mathrm{D}R}{\mathrm{D}b_p} &= \frac{\mathrm{D}}{\mathrm{D}b_p}[\varepsilon(\mathbf{p})]\int_{\Gamma_r}\left\{e_\mathrm{b}[T(\mathbf{r})] + \frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\
&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\
&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{\frac{\mathrm{D}}{\mathrm{D}b_p}[e_\mathrm{b}[T(\mathbf{r})]] + \frac{\mathrm{D}}{\mathrm{D}b_p}\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right.\\
&\quad\left.+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\frac{\mathrm{D}}{\mathrm{D}b_p}\tilde{q}_n^r(\mathbf{r})\right\}K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\
&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{e_\mathrm{b}[T(\mathbf{r})] + \frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\
&\quad\times\frac{\mathrm{D}}{\mathrm{D}b_p}[K(\mathbf{r},\mathbf{p})]\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\
&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{e_\mathrm{b}[T(\mathbf{r})] + \frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\
&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\frac{\mathrm{D}}{\mathrm{D}b_p}[\mathrm{d}\Gamma_r]\\
&\quad+\left[\frac{\mathrm{D}}{\mathrm{D}b_p}[\varepsilon(\mathbf{p})]e_\mathrm{b}(T^m) + \varepsilon(\mathbf{p})\frac{\mathrm{D}}{\mathrm{D}b_p}[e_\mathrm{b}(T^m)]\right]\\
&\quad\times\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})[1-\exp(-a|\mathbf{r}-\mathbf{p}|)]\,\mathrm{d}\Gamma_r + \varepsilon(\mathbf{p})e_\mathrm{b}(T^m)\\
&\quad\times\int_{\Gamma_r}\frac{\mathrm{D}}{\mathrm{D}b_p}[K(\mathbf{r},\mathbf{p})][1-\exp(-a|\mathbf{r}-\mathbf{p}|)]\,\mathrm{d}\Gamma_r\\
&\quad+\varepsilon(\mathbf{p})e_\mathrm{b}(T^m)\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})[1-\exp(-a|\mathbf{r}-\mathbf{p}|)]\frac{\mathrm{D}}{\mathrm{D}b_p}[\mathrm{d}\Gamma_r].
\end{aligned} \tag{9}$$

The main difficulty is to formulate the derivative of the kernel function with respect to design parameter. Kernel function depends on the geometry of the hole (see Eq. (3)) and can change during the shape modification. Consequently, we should discuss the most used domain modifications in order to investigate the effect of design parameters on the kernel function.

3.1. Translation

The angles Φ_p and Φ_r are design parameters independent (Fig. 4). Introducing the infinitesimal translation vector $\delta\boldsymbol{b}$, the transformation of the domain can be described for two-dimensional problems as follows

$$\mathbf{x}^* = \mathbf{x} + \delta\mathbf{b}, \tag{10}$$

where $\mathbf{x}$ and $\mathbf{x}^*$ denote the coordinates of an optional point before and after translation, respectively. The distance between the current point and the observation point can be determined, in view of Eq. (10) and Fig. 4, in the following form

$$\begin{aligned}
|\mathbf{r}^* - \mathbf{p}^*| &= \sqrt{(x_r^* - x_p^*)^2 + (y_r^* - y_p^*)^2}\\
&= \sqrt{(x_r - x_p)^2 + (y_r - y_p)^2} = |\mathbf{r} - \mathbf{p}|.
\end{aligned} \tag{11}$$

From Eq. (11) we have, that the kernel function is design parameter independent and the discussed material derivative is equal to zero $\mathrm{D}K/\mathrm{D}b_p = K_p = 0$.

3.2. Rotation

The angles Φ_p and Φ_r are design parameters independent (Fig. 5). Let us assume the two-dimensional problem and the rotation center situated on the same plane as the domain Ω. The infinitesimal rotation process can be stated by virtue of the infinitesimal rotation vector $\delta\boldsymbol{\omega}$ as follows [3]:

$$\mathbf{x}^* = \mathbf{x} + \delta\boldsymbol{\omega}\times\mathbf{x}. \tag{12}$$

The distance between the current point and the observation point is equal to

$$\begin{aligned}
|\mathbf{r}^* - \mathbf{p}^*| &= \sqrt{(x_r^* - x_p^*)^2 + (y_r^* - y_p^*)^2}\\
&= \sqrt{(x_r - x_p)^2 + (y_r - y_p)^2} = |\mathbf{r} - \mathbf{p}|.
\end{aligned} \tag{13}$$

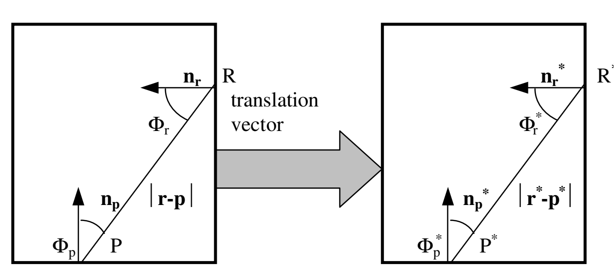

Fig. 4. Parameters of the kernel function for translation of the domain.

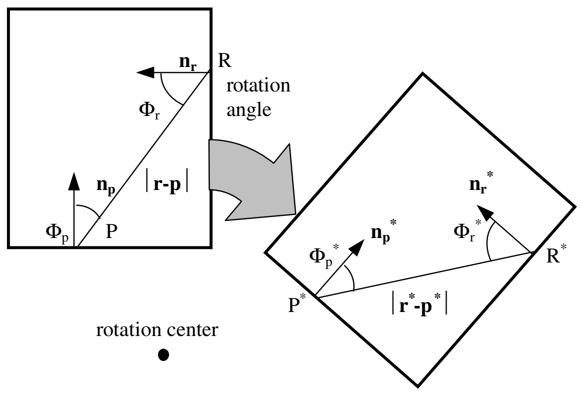

Fig. 5. Parameters of the kernel function for rotation of the domain.

It follows that the kernel function is design parameters independent and $\mathrm{D}K/\mathrm{D}b_p = K_p = 0$.

Introducing the Stefan–Boltzmann law, the material derivatives of the total hemispherical emissivity and the boundary element [5–7,10–13], as well as Eq. (7), we obtain from Eq. (9), after some simple transformations, the following expression:

$$\begin{aligned}\frac{\mathrm{D}R}{\mathrm{D}b_p} &= \varepsilon_p(\mathbf{p})\int_{\Gamma_r}\left\{\sigma T(\mathbf{r})^4+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{4\sigma T(\mathbf{r})^3[T^p(\mathbf{r})+\nabla T(\mathbf{r})\cdot\mathbf{v}^p]+\frac{1-\varepsilon_p(\mathbf{r})}{\varepsilon_p(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right.\\&\quad\left.+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}[(\tilde{q}_n^r)^p-\tilde{\mathbf{q}}_\Gamma^r\cdot\nabla_\Gamma v_n^p+\nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p+\tilde{q}_{n,n}^r v_n^p]\right\}\\&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{\sigma T(\mathbf{r})^4+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\&\quad\times\exp(-a|\mathbf{r}-\mathbf{p}|)[K_p(\mathbf{r},\mathbf{p})\\&\quad+K(\mathbf{r},\mathbf{p})(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)]\,\mathrm{d}\Gamma_r+\sigma(T^m)^3\\&\quad\times[\varepsilon_p(\mathbf{p})T^m+4\varepsilon(\mathbf{p})(T^m)^p]\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\sigma(T^m)^4\varepsilon(\mathbf{p})\int_{\Gamma_r}[1-\exp(-a|\mathbf{r}-\mathbf{p}|)]\\&\quad\times[K_p(\mathbf{r},\mathbf{p})+K(\mathbf{r},\mathbf{p})(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)]\,\mathrm{d}\Gamma_r.\end{aligned}\tag{14}$$

The radiation condition for the additional structure associated with the pth design parameter can be obtained by comparison of Eqs. (8) and (14) in the form

$$\begin{aligned}&(\tilde{q}_n^r)^p-\tilde{\mathbf{q}}_\Gamma^r\cdot\nabla_\Gamma v_n^p+\nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p+\tilde{q}_{n,n}^r v_n^p\\&\quad+\sigma T(\mathbf{p})^3[T(\mathbf{p})\varepsilon_p(\mathbf{p})+4T^p(\mathbf{p})+4\nabla T(\mathbf{p})\cdot\mathbf{v}^p]\\&\quad=\varepsilon_p(\mathbf{p})\int_{\Gamma_r}\left\{\sigma T(\mathbf{r})^4+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\\&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{4\sigma T(\mathbf{r})^3[T^p(\mathbf{r})+\nabla T(\mathbf{r})\cdot\mathbf{v}^p]\right.\\&\quad+\frac{1-\varepsilon_p(\mathbf{r})}{\varepsilon_p(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\\&\quad\times\left.\left[(\tilde{q}_n^r)^p-\tilde{\mathbf{q}}_\Gamma^r\cdot\nabla_\Gamma v_n^p+\nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p+\tilde{q}_{n,n}^r v_n^p\right]\right\}\\&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{\sigma T(\mathbf{r})^4+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}\exp(-a|\mathbf{r}-\mathbf{p}|)\\&\quad\times[K_p(\mathbf{r},\mathbf{p})+K(\mathbf{r},\mathbf{p})(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)]\,\mathrm{d}\Gamma_r\\&\quad+\sigma(T^m)^3[\varepsilon_p(\mathbf{p})T^m+4\varepsilon(\mathbf{p})(T^m)^p]\\&\quad\times\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r+\sigma(T^m)^4\varepsilon(\mathbf{p})\\&\quad\times\int_{\Gamma_r}[1-\exp(-a|\mathbf{r}-\mathbf{p}|)][K_p(\mathbf{r},\mathbf{p})+K(\mathbf{r},\mathbf{p})\\&\quad\times(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)]\,\mathrm{d}\Gamma_r.\end{aligned}\tag{15}$$

The left-hand side of above equation is defined by the vector-coordinate $\boldsymbol{p}$ at the observation point, and the right-hand side by the vector-coordinate $\boldsymbol{r}$ at the current point. The first three integrals on the right-hand side constitute the radiation of the walls of the convex hole. As well the material and the radiation properties of the walls, as the medium within the hole have the same properties for primary and additional structures. The last two integrals characterize the radiation within the medium and describe dissipation of the radiation within the hole.

The emissivity of the surface ε has different descriptions (cf. [15]). The performed analysis can be considerably simplified for the same radiation properties on the whole boundary of the convex hole. Thus, we assume that the total hemispherical emissivity is position independent $\varepsilon_p = \mathrm{D}\varepsilon/\mathrm{D}b_p = 0$; and the domain is translated or rotated. The radiation condition for the additional structure Eq. (15) can be transformed to the following:

$$\begin{aligned}&(\tilde{q}_n^r)^p-\tilde{\mathbf{q}}_\Gamma^r\cdot\nabla_\Gamma v_n^p+\nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p+\tilde{q}_{n,n}^r v_n^p\\&\quad+4\sigma T(\mathbf{p})^3[T^p(\mathbf{p})+\nabla T(\mathbf{p})\cdot\mathbf{v}^p]\\&\quad=\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{4\sigma T(\mathbf{r})^3[T^p(\mathbf{r})+\nabla T(r)\cdot\mathbf{v}^p]\right.\\&\qquad\left.+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}[(\tilde{q}_n^r)^p-\tilde{\mathbf{q}}_\Gamma^r\cdot\nabla_\Gamma v_n^p+\nabla_\Gamma\tilde{q}_n^r\cdot\mathbf{v}_\Gamma^p+\tilde{q}_{n,n}^r v_n^p]\right\}\\&\quad\times K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{\sigma T(\mathbf{r})^4+\frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^r(\mathbf{r})\right\}K(\mathbf{r},\mathbf{p})\\&\quad\times\exp(-a|\mathbf{r}-\mathbf{p}|)(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)\,\mathrm{d}\Gamma_r\\&\quad+4\varepsilon(\mathbf{p})(T^m)^p\sigma(T^m)^3\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r\\&\quad+\sigma(T^m)^4\varepsilon(\mathbf{p})\int_{\Gamma_r}K(\mathbf{r},\mathbf{p})\exp(-a|\mathbf{r}-\mathbf{p}|)\\&\quad\times(\mathrm{div}_\Gamma\mathbf{v}^p-2Hv_n^p)\,\mathrm{d}\Gamma_r;\quad p=1,2,\ldots,P.\end{aligned}\tag{16}$$

We can see at once, that the solution of Eq. (16) is more complicated in relation to the primary radiation condition Eq. (2). Both of them can be solved using the same method, for example the weighted residual method discussed by Bialecki [1] and Finlayson [8].

The external boundary Γ of the additional structure is composed from four portions $\Gamma = \Gamma_T \cup \Gamma_q \cup \Gamma_c \cup \Gamma_d$. Under the above assumption, the first-order sensitivity of an arbitrary behavioral functional F defined by Eq. (6) can be expressed according Dems and Rousselet [6]

$$F_P = \left[\int_\Omega \Psi_{,\dot T} T^p \,\mathrm{d}\Omega\right]_0^{t_f} + \int_0^{t_f} \left\{ \int_\Omega \left[\left(\Psi_{,T} - \frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{,\dot T})\right)T^p + \nabla_{\nabla T}\Psi \cdot \nabla T^p + \nabla_{\mathbf{q}}\Psi \cdot \mathbf{q}^p + \Psi_{,f} f^p\right]\mathrm{d}\Omega + \int_{\Gamma_T} [\gamma_{,T}(T_p^0 - \nabla_\Gamma T^0 \cdot \mathbf{v}_\Gamma^p - T_{,n}^0 v_n^p) + \gamma_{,q_n}(q_n^p - \mathbf{q}_\Gamma \cdot \nabla_\Gamma v_n^p)]\,\mathrm{d}\Gamma_T + \int_{\Gamma_q} [\gamma_{,T} T^p + \gamma_{,q_n}(q_{np}^0 - \nabla_\Gamma q_n^0 \cdot \mathbf{v}_\Gamma^p - q_{n,n}^0 \cdot v_n^p)]\,\mathrm{d}\Gamma_q + \int_{\Gamma_c} [\gamma_{,T} T^p + \gamma_{,q_n} h(T^p - T_\infty^p)]\,\mathrm{d}\Gamma_c + \int_{\Gamma_d} \gamma_{,T} T^p \,\mathrm{d}\Gamma_d + \int_\Gamma (\Psi + \gamma_{,n} - 2H\gamma) v_n^p \,\mathrm{d}\Gamma + \int_\Gamma \gamma_{,T_\infty} T_\infty^p \,\mathrm{d}\Gamma + \int_\Sigma]\gamma \mathbf{v}^p \cdot \mathbf{v}[\right\}\mathrm{d}t \quad p = 1,2,\ldots,P. \tag{17}$$

Analyzing the real problems, we conclude that γ satisfy often the following conditions

$$\gamma = \gamma(T, q_n) \quad \text{on } \Gamma_T \cup \Gamma_q; \quad \gamma = \gamma(T, T_\infty) \quad \text{on } \Gamma_c; \quad \gamma = \gamma(T) \quad \text{on } \Gamma_d. \tag{18}$$

Considering Eq. (18) and keeping in mind that the total derivatives T_p^0 on Γ_T and $(q_n^0)_p$ on Γ_q are known in advance, Eq. (17) can be rewritten in the form

$$F_P = \left[\int_\Omega \Psi_{,\dot T} T^p \,\mathrm{d}\Omega\right]_0^{t_f} + \int_0^{t_f} \left\{ \int_\Omega \left[\left(\Psi_{,T} - \frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{,\dot T})\right)T^p + \nabla_{\nabla T}\Psi \cdot \nabla T^p + \nabla_{\mathbf{q}}\Psi \cdot \mathbf{q}^p + \Psi_{,f} f^p\right]\mathrm{d}\Omega + \int_{\Gamma_T} [\gamma_{,T}(T_p^0 - \nabla_\Gamma T^0 \cdot \mathbf{v}_\Gamma^p - T_{,n}^0 v_n^p) + \gamma_{,q_n}(q_n^p - \mathbf{q}_\Gamma \cdot \nabla_\Gamma v_n^p)]\,\mathrm{d}\Gamma_T + \int_{\Gamma_q} [\gamma_{,T} T^p + \gamma_{,q_n}(q_{np}^0 - \nabla_\Gamma q_n^0 \cdot \mathbf{v}_\Gamma^p - q_{n,n}^0 \cdot v_n^p)]\,\mathrm{d}\Gamma_q + \int_{\Gamma_c} [\gamma_{,T} T^p + \gamma_{,T_\infty} T_\infty^p]\,\mathrm{d}\Gamma_c + \int_{\Gamma_d} \gamma_{,T} T^p \,\mathrm{d}\Gamma_d + \int_\Gamma (\Psi + \gamma_{,n} - 2H\gamma) v_n^p \,\mathrm{d}\Gamma + \int_\Sigma]\gamma \mathbf{v}^p \cdot \mathbf{v}[\right\}\mathrm{d}t \quad p = 1,2,\ldots,P. \tag{19}$$

Above sensitivity expression is a sum of integrals of time as well as within the domain Ω, on the whole external boundary Γ, the parts Γ_T, Γ_q, Γ_c, Γ_d and along the curve Σ between two adjacent parts of the piecewise smooth boundary Γ (cf. [6,13]). The thermal state fields of the additional structure T^p, $\mathbf{q}^p$ and q_n^p (i.e. the local sensitivities of the thermal state fields for the primary body) can be computed from additional heat conduction problems associated with the design parameter b_p. These problems are expressed by Eqs. (7), (15) and (16). For P shape design parameters, evaluation of the first-order sensitivity vector F_p by the direct method requires the solution of $P+1$ problems: i.e. the primary conduction problem and P additional conduction problems associated with each design parameter b_p.

4. Adjoint approach to sensitivity analysis

This alternative method for calculating the first-order sensitivity of the thermal functional Eq. (5) requires the solution of the one adjoint heat transfer problem and the primary heat conduction problem. Above adjoint and primary structures have the same shape, the thermal and the radiation properties (cf. Fig. 6). The adjoint method is most convenient for estimating the first-order sensitivities with respect to a few objective functionals. The governing equations of the adjoint heat conduction problem are the heat conduction equation within the structure and the boundary and initial conditions (cf. [6,13])

$$\begin{aligned} &\left.\begin{aligned} &-\mathrm{div}\mathbf{q}^a + f^a = c\tfrac{\mathrm{d}T^a}{\mathrm{d}\tau} \\ &\mathbf{q}^a = \mathbf{A} \cdot \nabla T^a + \mathbf{q}^{*a} \end{aligned}\right\} \quad \mathbf{x} \in \Omega; \\ &T^a(\mathbf{x},\tau) = T^{0a}(\mathbf{x},\tau) \quad \mathbf{x} \in \Gamma_T; \\ &q_n^a(\mathbf{x},\tau) = \mathbf{n}\cdot\mathbf{q}^a = q_n^{0a}(\mathbf{x},\tau) \quad \mathbf{x} \in \Gamma_q; \\ &q_n^a(\mathbf{x},\tau) = \mathbf{n}\cdot\mathbf{q}^a = h\lfloor T^a(\mathbf{x},\tau) - T_\infty^a(\mathbf{x},\tau)\rfloor \quad \mathbf{x} \in \Gamma_c; \\ &\mathbf{n}\cdot\mathbf{q}^{ar} = q_n^{ar}(\mathbf{x},\tau) \quad \mathbf{x} \in \Gamma_d; \\ &\mathbf{n}\cdot\tilde{\mathbf{q}}^{ar} = \tilde{q}_n^{ar}(\mathbf{x},\tau) \quad \mathbf{x} \in \Gamma_r; \\ &\mathbf{T}^a(\mathbf{x},\tau=\mathbf{0}) = \mathbf{T}_\mathbf{0}^a(\mathbf{x},\tau=\mathbf{0}) \quad \mathbf{x} \in (\Omega \cup \Gamma). \end{aligned} \tag{20}$$

The radiative heat flux density $\tilde{q}_n^{ar}$ on the internal boundary Γ_r is the solution of the radiation condition for adjoint structure. This condition is due to Bialecki [1] and has the form very similar to the condition for the primary problem, namely

$$\tilde{q}_n^{ar}(\mathbf{p}) + \varepsilon(\mathbf{p})e_b[T^a(\mathbf{p})] = \varepsilon(\mathbf{p})\int_{\Gamma_r}\left\{e_b[T^a(\mathbf{r})] + \frac{1-\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\tilde{q}_n^{ar}(\mathbf{r})\right\}K(\mathbf{r},\mathbf{p}) \times \exp(-a|\mathbf{r}-\mathbf{p}|)\,\mathrm{d}\Gamma_r + \varepsilon(\mathbf{p})e_b(T^m) \times \int_{\Gamma_r} K(\mathbf{r},\mathbf{p})[1 - \exp(-a|\mathbf{r}-\mathbf{p}|)]\,\mathrm{d}\Gamma_r \tag{21}$$

The left-hand side of Eq. (21) is calculated at the observation point defined by the vector $\boldsymbol{p}$, whereas the right-hand side at the current point defined by the vector $\boldsymbol{r}$. The total

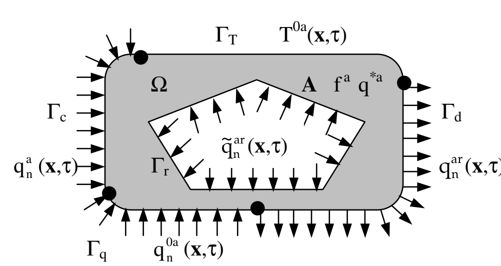

Fig. 6. Adjoint heat conduction problem.

heat flux for the adjoint structure is stated using the flux of radiant emission of the wall, the blackbody emissive power and the emissivity of the surface. The first term on the right-hand side of Eq. (21) describes the radiant energy of the wall. The second term on the right-hand side has the same form as for primary structure (cf. Eq. (2)) and describes the radiation of the isothermal and participating in radiating process medium within the hole.

Our next objective is to formulate the conditions for adjoint structure using the heat conduction equation Eq. (20). We follow [4] in considering the following identity:

$$\begin{aligned}
&\left[\int_\Omega cT^{\mathrm a}T^p\,\mathrm d\Omega\right]_{t=t_{\mathrm f}} + \int_0^{t_{\mathrm f}}\int_\Omega (T^p f^{\mathrm a} + \nabla T^p\cdot\mathbf q^{*a})\,\mathrm d\Omega\,\mathrm dt\\
&\quad + \int_0^{t_{\mathrm f}}\int_{\Gamma_T} T^{0\mathrm a}q_n^p\,\mathrm d\Gamma_T\,\mathrm dt - \int_0^{t_{\mathrm f}}\int_{\Gamma_q} T^p q_n^{0\mathrm a}\,\mathrm d\Gamma_q\,\mathrm dt\\
&\quad - \int_0^{t_{\mathrm f}}\int_{\Gamma_d} T^p q_n^{\mathrm{ar}}\,\mathrm d\Gamma_d\,\mathrm dt + \int_0^{t_{\mathrm f}}\int_{\Gamma_c} hT^pT^{\mathrm a}_\infty\,\mathrm d\Gamma_c\,\mathrm dt\\
&\quad - \int_0^{t_{\mathrm f}}\int_\Omega cT^p\frac{\mathrm dT^{\mathrm a}}{\mathrm dt}\,\mathrm d\Omega\,\mathrm dt - \int_0^{t_{\mathrm f}}\int_\Omega cT^p\frac{\mathrm dT^{\mathrm a}}{\mathrm d\tau}\,\mathrm d\Omega\,\mathrm dt\\
&= \left[\int_\Omega cT^{\mathrm a}T^p\,\mathrm d\Omega\right]_{t=0} + \int_0^{t_{\mathrm f}}\int_\Omega (T^{\mathrm a}f^p + \nabla T^{\mathrm a}\cdot\mathbf q^{*p})\,\mathrm d\Omega\,\mathrm dt\\
&\quad + \int_0^{t_{\mathrm f}}\int_{\Gamma_T} T^{0p}q_n^{\mathrm a}\,\mathrm d\Gamma_T\,\mathrm dt - \int_0^{t_{\mathrm f}}\int_{\Gamma_q} T^{\mathrm a}q_n^{0p}\,\mathrm d\Gamma_q\,\mathrm dt\\
&\quad - \int_0^{t_{\mathrm f}}\int_{\Gamma_d} T^{\mathrm a}q_n^{rp}\,\mathrm d\Gamma_q\,\mathrm dt\\
&\quad + \int_0^{t_{\mathrm f}}\int_{\Gamma_c} T^{\mathrm a}(hT^p_\infty - \mathbf q_\Gamma\cdot\nabla_\Gamma v_n^p)\,\mathrm d\Gamma_c\,\mathrm dt. \qquad (22)
\end{aligned}$$

Thus, the following transformation rule between the time t of primary and additional problems and the time τ of adjoint problem is considered

$$\tau = t_{\mathrm f} - t;\quad t = t_{\mathrm f}\quad \Rightarrow \tau = 0;\quad t = 0 \Rightarrow \tau = t_{\mathrm f} \qquad (23)$$

It follows, that the final time $t = t_{\mathrm f}$ at the primary and additional problem is equivalent to the starting time at the adjoint problem. Consequently, the derivatives of temperature are equal to $-\frac{\mathrm dT^{\mathrm a}}{\mathrm d\tau} = \frac{\mathrm dT^{\mathrm a}}{\mathrm dt}$. Considering Eq. (22) into the right-hand side of Eq. (17), we can observe the vanishing of some sum of integrals, if the following conditions are satisfy

$$\begin{aligned}
&T^{\mathrm a}(\mathbf x,\tau=0) = \frac1c\Psi_{,\dot T}(\mathbf x, t=t_{\mathrm f})\quad \mathbf x\in(\Omega\cup\Gamma);\\
&f^{\mathrm a}(\mathbf x,\tau) = \Psi_{,T}(\mathbf x,t) - \frac{\mathrm d}{\mathrm dt}\Psi_{,T}(\mathbf x,t)\quad \mathbf x\in\Omega;\\
&\mathbf q^{*a}(\mathbf x,\tau) = \nabla_{\nabla T}\Psi(\mathbf x,t) + \nabla_{\mathbf q}\Psi(\mathbf x,t)\cdot\mathbf A(\mathbf x)\quad \mathbf x\in\Omega;\\
&T^{0\mathrm a}(\mathbf x,\tau) = \gamma_{,q_n}(\mathbf x,t)\quad \mathbf x\in\Gamma_T;\\
&q_n^{0\mathrm a}(\mathbf x,\tau) = -\gamma_{,T}(\mathbf x,t)\quad \mathbf x\in\Gamma_q;\\
&T^{\mathrm a}_\infty(\mathbf x,\tau) = \frac1h\gamma_{,T}(\mathbf x,t) + \gamma_{,q_n}(\mathbf x,t)\quad \mathbf x\in\Gamma_c;\\
&q_n^{\mathrm{ar}}(\mathbf x,\tau) = \sigma[T^{\mathrm a}(\mathbf x,\tau)]^4;\\
&T^{\mathrm a}(\mathbf x,\tau) = \left[\frac{-\gamma_{,T}(\mathbf x,t)}{\sigma}\right]^{0.25}\quad \mathbf x\in\Gamma_d.
\end{aligned} \qquad (24)$$

The first-order sensitivity expression Eq. (17) can be transformed using Eq. (24) to the form

$$\begin{aligned}
F_p = &-\left[\int_\Omega(\Psi_{,\dot T} - cT^{\mathrm a})(T_p - \nabla T\cdot\mathbf v^p)\,\mathrm d\Omega\right]_{t=0}\\
&+\int_0^{t_{\mathrm f}}\left\{\int_\Omega\left[(\nabla_q\Psi + \nabla T^{\mathrm a})\cdot\mathbf q^{*p} + (\Psi_{,\mathrm f} + T^{\mathrm a})f^p\right]\mathrm d\Omega\right.\\
&+\int_{\Gamma_T}\left[(\gamma_{,T} + q_n^{\mathrm a})(T_p^0 - \nabla_\Gamma T^0\cdot\mathbf v_\Gamma^p - T^0_{,n}v_n^p)\right.\\
&\left.-\gamma_{,q_n}\mathbf q_\Gamma\cdot\nabla_\Gamma v_n^p\right]\mathrm d\Gamma_T\\
&+\int_{\Gamma_q}\left[(\gamma_{,q_n} - T^{\mathrm a})(q^0_{np} - \nabla_\Gamma q_n^0\cdot\mathbf v_\Gamma^p - q^0_{n,n}v_n^p)\right.\\
&\left.-T^{\mathrm a}\mathbf q_\Gamma^0\cdot\nabla_\Gamma v_n^p\right]\mathrm d\Gamma_q\\
&+\int_{\Gamma_c}\left[T^{\mathrm a}hT^p_\infty - T^{\mathrm a}\mathbf q_\Gamma\nabla_\Gamma v_n^p - \gamma_{,q_n}hT^p_\infty\right]\mathrm d\Gamma_c\\
&-\int_{\Gamma_d}T^{\mathrm a}q_n^{rp}\,\mathrm d\Gamma_d + \int_\Gamma(\Psi + \gamma_{,n} - 2H\gamma)v_n^p\,\mathrm d\Gamma\\
&\left.+\int_\Gamma\gamma_{,T_\infty}T^p_\infty\,\mathrm d\Gamma + \int_\Sigma]\gamma\mathbf v^p\cdot\mathbf{\upsilon}[\right\}\mathrm dt. \qquad (25)
\end{aligned}$$

Introducing the conditions stated by Eq. (18), we simplify the first-order sensitivity expression to the following:

$$\begin{aligned}
F_p = &-\left[\int_\Omega(\Psi_{,\dot T} - cT^{\mathrm a})(T_p - \nabla T\cdot\mathbf v^p)\,\mathrm d\Omega\right]_{t=0}\\
&+\int_0^{t_{\mathrm f}}\left\{\int_\Omega\left[(\nabla_q\Psi + \nabla T^{\mathrm a})\cdot\mathbf q^{*p} + (\Psi_{,\mathrm f} + T^{\mathrm a})f^p\right]\mathrm d\Omega\right.\\
&+\int_{\Gamma_T}\left[(\gamma_{,T} + q_n^{\mathrm a})(T_p^0 - \nabla_\Gamma T^0\cdot\mathbf v_\Gamma^p - T^0_{,n}v_n^p)\right.\\
&\left.-\gamma_{,q_n}\mathbf q_\Gamma\cdot\nabla_\Gamma v_n^p\right]\mathrm d\Gamma_T\\
&+\int_{\Gamma_q}\left[(\gamma_{,q_n} - T^{\mathrm a})(q^0_{np} - \nabla_\Gamma q_n^0\cdot\mathbf v_\Gamma^p - q^0_{n,n}v_n^p)\right.\\
&\left.-T^{\mathrm a}\mathbf q_\Gamma^0\cdot\nabla_\Gamma v_n^p\right]\mathrm d\Gamma_q\\
&+\int_{\Gamma_c}\left[(T^{\mathrm a}h + \gamma_{,T_\infty})T^p_\infty - T^{\mathrm a}\mathbf q_\Gamma\nabla_\Gamma v_n^p\right]\mathrm d\Gamma_c\\
&\left.-\int_{\Gamma_d}T^{\mathrm a}q_n^{rp}\,\mathrm d\Gamma_d + \int_\Gamma(\Psi + \gamma_{,n} - 2H\gamma)v_n^p\,\mathrm d\Gamma + \int_\Sigma]\gamma\mathbf v^p\cdot\mathbf{\upsilon}[\right\}\mathrm dt. \qquad (26)
\end{aligned}$$

Similarly as the sensitivity expression for the direct approach, above expression is a sum of integrals of time as well as within the domain Ω, on the external boundary Γ, its portions Γ_T, Γ_q, Γ_c, Γ_d and along the discontinuity curve Σ. The thermal state fields characterizing the adjoint structure (i.e. $T^{\mathrm a}$, $q^{\mathrm a}$, $q_n^{\mathrm a}$) are obtained from Eqs. (20), (21) and (24). Summarizing, formulation of the first-order sensitivity vector F_p requires now the solution of $N+1$ problems if the number of functionals is equal to N, as is easy to check.

5. Shape optimization problem

By the shape optimization we mean the minimization of the objective functional with the imposed constraint on the structural cost C. Assuming the homogeneous structure in real technical problems, the structural cost is in fact proportional to the area of domain Ω. Thus, for the unit cost u we can denote this problem as follows

$$\text{Minimize} \quad F \text{ or minimize } (-F)$$

$$\text{Subject to} \quad C - C_0 = \int_\Omega u\, \mathrm{d}\Omega - C_0 \leqslant 0, \tag{27}$$

We may introduce the Lagrange functional given in the form

$$F' = F + \chi(C - C_0 + \xi^2) \tag{28}$$

The physical interpretation of the variable ξ^2 is given for example in [5]. Following the stationarity of Lagrange functional Eq. (28), the following optimality conditions are stated

$$\begin{cases} \frac{\mathrm{D}F}{\mathrm{D}b_p} = -\chi \int_\Omega u v_n^p\, \mathrm{d}\Gamma, \\ \int_\Omega u\, \mathrm{d}\Omega - C_0 + \xi^2 = 0. \end{cases} \tag{29}$$

In order to solve these optimality conditions we should determine the first-order sensitivities using the direct and adjoint approaches represented by Eqs. (17), (19), (25) and (26), respectively. Consequently, the objective functional should be next stated. Dems and Korycki [4] present some most used objective functionals. The first cited form is the following:

$$F = \int_\Gamma q_n\, \mathrm{d}\Gamma; \quad \Gamma \in \Gamma_{\text{external}} \tag{30}$$

Minimization of above functional corresponds to design of optimal heat isolator, whereas for a model of heat radiator the functional Eq. (30) should be maximized

$$F = \int_\Omega f\, \mathrm{d}\Omega \tag{31}$$

The form of functional is associated with the amount of heat generated within the structural domain. Thus, the optimal shape can be considered from the point of minimizing or maximizing of Eq. (31).

$$F = \left[\int_\Gamma \left(\frac{T}{T_0}\right)^n \mathrm{d}\Gamma\right]^{\frac{1}{n}}; \quad n \to \infty \quad \Gamma \in \Gamma_{\text{external}} \tag{32}$$

The functional is now the global measure of maximum local temperature within the domain and can help to determine the optimal shape of structural boundary minimizing the temperature distribution within the structure, respectively.

6. Numerical example

The discussed expressions can be applied to the two-dimensional shape optimization of the isolated inlet channel with a steam (Fig. 7). Let us assume the thermal orthotropic material of the thermal conductivity matrix equal to

$$\mathbf{A} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} 0.042 & 0 \\ 0 & 0.032 \end{vmatrix} \left[\frac{\mathrm{W}}{\mathrm{m\,K}}\right] \tag{33}$$

Upper part of the filling has the prescribed temperature changed in time according the function $T_0 = 390 + \exp(0.01t)$ [K] . The calculation were performed for $t_0 = 0$, $t_k = 240$ s, $\Delta t = 60$ s. Thus, the temperature changes from the initial value $T = 391$ K to the final value $T = 401.023176$ K. All other parts of the boundary are thermally isolated and consequently the heat flux density is equal to zero $q_n = 0$ on these portions. The gray and diffuse walls of the channel have the prescribed constant values of the temperature $T = 340$ K and the surface emissivity $e = 0.15$. The steam within the channel is an isothermal, participating medium of the constant absorption coefficient $a = 0.20$.

Let us assume that the heat generation source and the initial heat flux density are equal to zero ($f = 0, \boldsymbol{q}^* = 0$). The primary problem can be introduced in view of Eq. (1) as follows:

$$\left.\begin{aligned} -\operatorname{div}\mathbf{q} &= c\tfrac{\mathrm{d}T}{\mathrm{d}t} \\ q &= \mathbf{A}\cdot\nabla T \end{aligned}\right\} \quad \mathbf{x} \in \Omega; \\ T = T^0 = 390 + \exp(0.01t) \text{ [K]} \quad \text{on } \Gamma_T, \\ q_n = 0 \quad \text{on } \Gamma_\mathrm{q}, \\ \mathbf{n}\cdot\tilde{\mathbf{q}}^r(\mathbf{x}) = \tilde{q}_n^r \quad \mathbf{x} \in \Gamma_\mathrm{r}, \\ T = T_0 \quad \text{on } \Omega \cup \Gamma \tag{34}$$

The radiative heat flux density is the solution of the condition according Bialecki, cf. Eq. (2). The shape optimization is a simple engineering problem of structural heat isolator with imposed equality constraint on the cost in the form (cf. Eq. (30) and Fig. 7a)

$$F = \int_{\Gamma_T} q_n\, \mathrm{d}\Gamma_T \to \text{ min subject to } C - C_0 = 0. \tag{35}$$

Let us compose the external boundary using the 7 piecewise linear portions, the main curvatures of the boundary are $H \to 0$.

First we discuss the direct approach to sensitivity analysis. Using as well Eqs. (1), (7), (35), as assuming the material derivatives $(q_n^0)_p$ on Γ_q, $(T^0)_p$ on Γ_T, $(T_0)_p$ on $\Omega \cup \Gamma$ as known in advance, the additional structure can be described by following equations:

$$\left.\begin{aligned} -\operatorname{div}\mathbf{q}^p &= c\tfrac{\mathrm{d}T^p}{\mathrm{d}t} \\ \mathbf{q}^p &= \mathbf{A}\cdot\nabla T^p \end{aligned}\right\} \quad \mathbf{x} \in \Omega; \\ q_n^p(\mathbf{x},t) = 0 \quad \mathbf{x} \in \Gamma_q, \\ T^p(\mathbf{x},t) = -\nabla T^0\cdot\mathbf{v}^p \quad \mathbf{x} \in \Gamma_T, \\ \mathbf{n}\cdot\tilde{q}^p = \tilde{q}_n^{rp} \quad \mathbf{x} \in \Gamma_r, \\ T_0^p(\mathbf{x},0) = -\nabla T_0\cdot\mathbf{v}^p \quad \mathbf{x} \in (\Omega \cup \Gamma). \tag{36}$$

To obtain the heat flux density $\tilde{q}_n^{rp}$ normal to the boundary portion Γ_r we introduce the total hemispherical emissivity

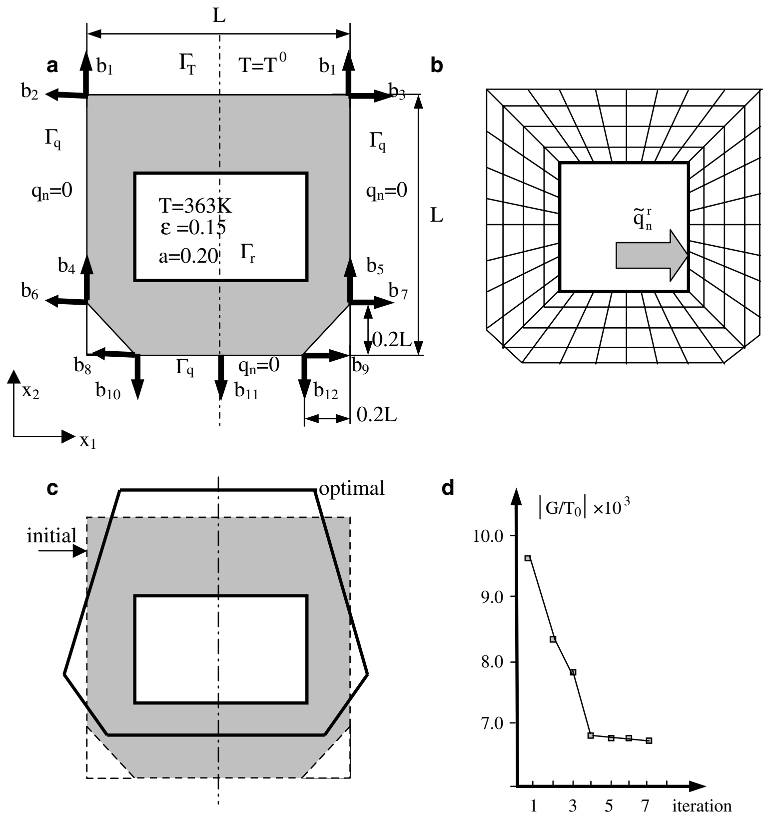

Fig. 7. Shape optimal design of isolated channel with a steam inside: (a) design parameters and boundary conditions, (b) FEM-net, (c) initial and optimal shapes, (d) optimization history.

as position independent and should solve the radiation equation Eq. (16) for the additional structure. Next, the first-order sensitivity vector can be determined by the simple adaptation of Eq. (17).

Our next goal is to determine the equations for adjoint approach to sensitivity analysis. Making use of Eqs. (20), (24) and (35), we formulate this problem as follows:

$$\begin{array}{l}
\left.\begin{array}{l}\operatorname{div}\mathbf{q}^{\mathrm{a}} = c\frac{\mathrm{d}T^{a}}{\mathrm{d}\tau}\\ \mathbf{q}^{\mathrm{a}} = \mathbf{A}\cdot\nabla T^{\mathrm{a}}\end{array}\right\} \quad \mathbf{x}\in\Omega;\\
T^{\mathrm{a}}(\mathbf{x},\tau) = T^{0\mathrm{a}}(\mathbf{x},\tau) \quad \mathbf{x}\in\Gamma_T;\\
q_n^{\mathrm{a}}(\mathbf{x},\tau) = \mathbf{n}\cdot\mathbf{q}^{\mathrm{a}} = q_n^{0\mathrm{a}}(\mathbf{x},\tau) \quad \mathbf{x}\in\Gamma_q;\\
\mathbf{n}\cdot\tilde{\mathbf{q}}^{\mathrm{a}r} = \tilde{q}_n^{\mathrm{a}r}(\mathbf{x},\tau) \quad \mathbf{x}\in\Gamma_r;\\
T^{\mathrm{a}}(\mathbf{x},\tau=0) = T_0^{\mathrm{a}}(\mathbf{x},\tau=0) \quad \mathbf{x}\in(\Omega\cup\Gamma);\\
T^{\mathrm{a}}(\mathbf{x},\tau=0) = 0 \quad \mathbf{x}\in(\Omega\cup\Gamma);\\
f^{\mathrm{a}}(\mathbf{x},\tau) = 0 \quad \mathbf{x}\in\Omega;\\
f^{\mathrm{a}}(\mathbf{x},\tau) = 0 \quad \mathbf{x}\in\Omega;\\
\mathbf{q}^{*a}(\mathbf{x},\tau) = 0 \quad \mathbf{x}\in\Omega;\\
T^{0\mathrm{a}}(\mathbf{x},\tau) = 1 \quad \mathbf{x}\in\Gamma_T;\\
q_n^{0\mathrm{a}}(\mathbf{x},\tau) = 0 \quad \mathbf{x}\in\Gamma_q.
\end{array} \tag{37}$$

The radiative heat flux density $\tilde{q}_n^{ar}$ has the simple form described by Eq. (21). The radiation equations for primary, additional and adjoint structures were solved using the weighted residual method (cf. [8]).

The independent design parameters are 12 coordinates of the selected points on the external boundary, which are depicted by the arrows b_1–b_{12} on Fig. 7a. The upper part of the external boundary contact the floor. Consequently, this part of the boundary has the same number of design variable, i.e. the same vertical coordinate during the optimization process.

The analysis step of the optimization procedure was performed using the Finite Element Method. The structure domain was discretized using the 4-nodal elements net (cf. Fig. 7b). The solution procedure is iterative. At each step we should solve first the primary problem, next the additional problems or the adjoint problem, respectively, and the obtained results are necessary sensitivities. At the synthesis stage the second-order Newton procedure and the method of steepest descent are applied alternatively to find the directional minimum. The initial and the optimal shapes of the structure are shown in Fig. 7c. It can be seen from Fig. 7c, that the optimal boundary is located far from the hole with isothermal and participating medium. The history of optimization process is shown briefly in Fig. 7d, where the changes of objective functional are plotted in terms of iteration number.

Let us next rotate the hole with isothermal and participating medium and the rotation center is located on the lower part of the filling (Fig. 8b). The external boundary of the filling is now stationary. Thus, we have now only

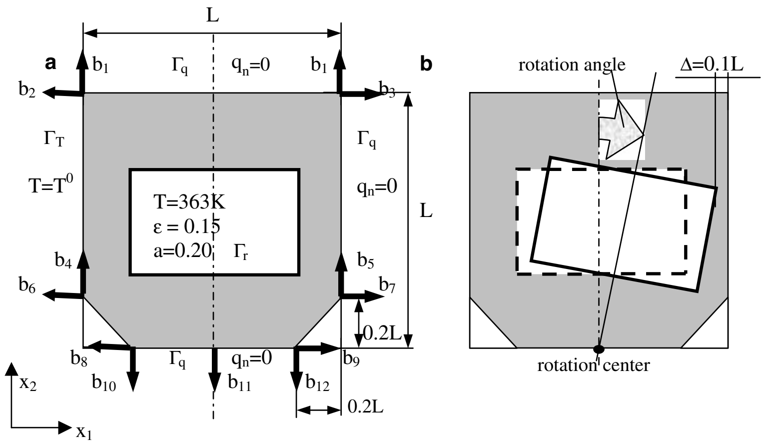

Fig. 8. Shape optimal design of isolated channel with a steam inside: (a) design parameters and boundary conditions, (b) initial and optimal shapes of the rotated hole.

one design parameter, which is the rotation angle. We assume that the left-hand side of the isolation has the prescribed temperature. All other portions of the external boundary are thermally isolated and the heat flux density is equal to zero $q_n = 0$ (see Fig. 8a). The additional constraint is in this case the minimal dimension of the external isolation, which is equal to $\Delta = 0.1L$ (see Fig. 8b). The initial and optimal location of the hole for the same objective functional as before, cf. Eq. (35), are depicted now on the Fig. 8. The boundary portion Γ_T of the optimal structure is located so far as possible from the hole with an isothermal and participating stream inside. The heat flux through the left-hand side of external boundary is consequently minimized and the optimal solution was improved with 10.21% in comparison to the initial value, in nine iterations.

7. Conclusion remarks

The aim of this paper was to present the application of direct and adjoint approaches to sensitivity analysis in the shape optimization problems with radiation on the internal and external boundary portions. The first-order sensitivity vectors were formulated using as well the material derivative concept as direct and adjoint approaches to sensitivity analysis. The direct method is convenient for the low number of functionals, because the number of additional solutions is equal to number of functionals. The disadvantage is in this case the complicated form of the integral radiation condition. The problem is easier to solve for the simplified form of equation (i.e. position independent radiation properties and stationary shape of the hole). The adjoint method is useful for the low number of objective functionals, but is requires the transformation of time. The radiation equation has the same form as for the primary structure and is more convenient to calculate the radiative heat flux density.

We introduce also an effective tool for generating the optimal boundary shapes for a wide class of design and redesign problems with radiative heat transfer. Thus, the obtained first-order sensitivity expressions can be also applied to solve the shape identification problems, respectively. Obtained results can be verified by differential numerical methods. The detailed analysis of such implementations and their efficiency and accuracy is beyond the scope of this paper and will be studied in details in consecutive paper.

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